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A survey: From a surgical view of Alexander invariants

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1 Abstract

The Alexander polynomial is an effective knot-invariant still now. Levine and Rolfsen introduced a surgical view of Alexander invariants. In this note, we will study the surgical view and its applications: unknotting number and knot adjacency.

2 Surgical description

The Alexander polynomial was introduced by Alexander [1] in 1928. Since then, several knot theorists have introduced alternative definitions of Alexander polynomial: Seifert [18] in 1934, Fox [3] in 1953), Levine [8] in 1965, and so on.

Their definitions are based on the infinite cyclic covering space of the complement of a given knot. Let K be a knot in the 3-sphere S^3, $X = S^3 \setminus K$, $\widetilde{X}_\infty$ the infinite cycle covering space of X. For the Laurent polynomial ring $\Lambda = \mathbf{Z}[t, t^{-1}]$, $H_1(\widetilde{X}_\infty)$ is regarded as a Λ-module, which is called the *Alexander invariant* of K. Let M be a presentation matrix of $H_1(\widetilde{X_\infty})$. Then $\Delta_K(t) = \det M$ is called the *Alexander polynomial* of K.

We need the following fact.

Proposition 1 ([21]). *For a diagram of a knot, certain crossing changes yield a diagram of a trivial knot.*

From Proposition 1, We have Proposition 2, that is called a *surgical description* ([15], [16]) of a knot.

Proposition 2 ([15], [16]). *Let K be a knot, and K_0 a trivial knot. Then, there exsist solid tori $T_1, \dots T_n$ in $S^3 \setminus K_0$, and a homeomorphism $\varphi : S^3 \setminus K_0 \to S^3 \setminus K_0$ such that*
(1) $\varphi(K_0) = K$,
(2) *the core of* $T_1 \cup \cdots \cup T_n$ *are trivial,*
(3) $\mathrm{lk}(T_i, K_0) = \mathrm{lk}(\varphi(T_i), K) = 0$ $(\forall i)$, *and*
(4) $\mathrm{lk}(\mu_i', T_i) = \pm 1$, *where* μ_i *a meridian of* $\varphi(T_i)$ *and* $\mu_i' = \varphi^{-1}(\mu_i)$.

We can construct a Seifert surface of K missing $T_1 \cup \cdots \cup T_n$ by the condition $\mathrm{lk}(T_i, K_0) = 0$. Cut along the Seifert surface and make an infinite number of copies. Paste them along opening sections one after another, and we have the infinite cyclic covering space $\widetilde{X_\infty}$ of $X = S^3 \setminus K$. Reading the linking numbers of tori, we have an Alexander matrix and the Alexander polynomial as follows:

Key Proposition 3 ([8], [15], [16]). *Let K be a knot. Then, K has an Alexander matrix $M = (m_{ij}(t))$ of the form: (1) $m_{ij}(t) = m_{ji}(t^{-1})$, and (2) $|m_{ij}(1)| = \delta_{ij}$, where $\delta_{ij} = 1$ (if $i = j$), 0 (if $i \neq j$). The converse is also valid.*

3 Unknotting number.

For a knot K, the *unknotting number* ([21]) of K, denoted by $u(K)$, is defined to be the minimum number of crossing changes which yield a diagram of a trivial knot among all diagrams of K. In surgical description of K, the minimum number of solid tori $T_1 \cup \cdots \cup T_n$ is called the *surgical description number* of K, denoted by $sd(K)$. The minimum size of presentation matrices of $H_1(\widetilde{X_\infty})$ is denoted by $m(K)$.

Proposition 4 ([9]) . $0 \leq m(K) \leq sd(K) \leq u(K)$.

Proposition 5 ([14], [19], [10]). *Let K be the knot 5_1 (or, 7_4, 10_{106}, 10_{109}, 10_{121}). We have $sd(K) = u(K) = 2$.*

Sketch of Proof. Let K be the knot 5_1. A crossing change yields a diagram of 3_1. We would suppose that $sd(K) = 1$. Then, 3_1 had an Alexander matrix of the form $M = \begin{pmatrix} \Delta_K(t) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$ with $m(t) = m(t^{-1})$, $|m(1)| = 1$, and $r(1) = 0$. Put $t = -1$ on $\det M = \pm(t - 1 + t^{-1})$, and we had $\begin{vmatrix} \Delta_K(-1) & r(-1) \\ r(-1) & m(-1) \end{vmatrix} = \pm 3$. We had $r(-1)^2 \equiv \pm 3 \pmod 5$, a contradiction.

Remark. In [10], there are mistakes for 10_{83} and 10_{117}. So we omit them from Proposition 5. The author would like to thank Professor Kanenobu for his pointing out.

4 Knot adjacency.

For knots J and K, if J is obtained from K by a single crossing change, J is said to be *adjacent* to K. The unknotting number one knot is a knot which is adjacent to a trivial knot.

The Alexander polynomials of unknotting number one knots are characterized as follows.

Theorem 6 ([7], [17]). *The Alexander polynomials $\Delta_K(t)$ of the unknotting number one knots are characterized by* (1) $\Delta_K(t^{-1}) = \Delta_K(t)$, *and* (2) $|\Delta_K(1)| = 1$.

The Alexander polynomials of knots which are obtained from the trefoil knot by a single crossing change are characterized as follows.

Theorem 7 ([11]). *The Alexander polynomials $\Delta_K(t)$ of the knots which are adjacent to a trefoil knot are characterized by* (1) $\Delta_K(t^{-1}) = \Delta_K(t)$, (2) $|\Delta_K(1)| = 1$, *and* (3) $|\Delta_K(\zeta)| = 0, 1,$ *or* $p_1^{e_1} \cdots p_n^{e_n}$ *for a complex ζ with $\zeta^2 - \zeta + 1 = 0$ where p_i is prime, e_i is even for $p_i = 2, 3k+2$, and e_j is arbitrary for $p_j = 3, 3k+1$.*

Remark. Such integers are $N = 0, 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, \ldots$.

Sketch of Proof. It is sufficient to show (3). Let J be a knot obtained from a trefoil knot by a single crossing change. Then, it can be seen that $\Delta_J(t)$ is equal to the determinant of $\begin{pmatrix} \pm(-t+1-t^{-1}) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$ up to sign. Put $t = \zeta$, $|\Delta_J(\zeta)| = |-r(\zeta)r(\zeta^{-1})|$. There exist integers a and b such that $r(\zeta) = a\zeta + b$.
$|-r(\zeta)r(\zeta^{-1})| = |(a\zeta + b)(a\zeta^{-1} + b)| = |a^2 + b^2 - ab|$.

By a standard argument in Number Theory (cf. [5], [20]), $|a^2 + b^2 - ab|$ is written as $0, 1$, or $p_1^{e_1} \cdots p_n^{e_n}$ where p_i is prime, e_i is even for $p_i = 2, 3k+2$, and e_j is arbitrary for $p_j = 3, 3k+1$.

The converse is a bit hard to show, so we omit it here.

The above type theorem can be shown for knots whose Alexander polynomials are monic (cf. [13]).

5 n-adjacency.

Let J and K be knots. If J has a diagram containing n crossings such that crossing changes any $0 < m \leq n$ of them yield a diagram of K, J is said to be *n-adjacent* ([2]) (or *strongly $(n-1)$-similar* ([4])) to K.

Proposition 8. ([Stanford (cf. [6])]) *Let J and K be knots. If J is 2-adjacent to K, then $|a_2(J) - a_2(K)| \leq 1$, where a_2 is the coefficient of z^2 in the Conway polynomial.*

Sketch of Proof. For a certain diagram D of J, there exist two crossings c_1 and c_2 such that crossing changes any non-empty subset of them yield a diagram of K. Let D_1 be the diagram from D by crossing change at c_1, D_2 the diagram from D by crossing change at c_2, and D_3 the diagram from D by crossing change at c_1, c_2. Let S_1 be the diagram from D by smoothing at c_1, and S_2 the diagram from D_2 by smoothing at c_1. Let ε be the sign of c_1. By the skein relation, we have

$$\nabla_D(z) - \nabla_{D_1}(z) = -\varepsilon z \nabla_{S_1}(z),$$

$$\nabla_{D_2}(z) - \nabla_{D_3}(z) = -\varepsilon z \nabla_{S_2}(z).$$

Since S_1 and S_2 differ only by c_2, we have $|\mathrm{lk}(S_1) - \mathrm{lk}(S_2)| = 1$.

Since D_1, D_2, and D_3 are diagrams of the same K, $|a_2(J) - a_2(K)| \leq 1$.

Proposition 9 ([12]). *Let K be 2-adjacent to a trivial knot. Then, the Alexander polynomial of K is equal to $\pm 1 - r(t)r(t^{-1})$, where $r(t) = c_1(t-1) + c_2(t-1)^2 + \cdots + c_n(t-1)^n$ with $c_1 = 0, \pm 1$. The converse is also valid.*

The proof of Proposition 9 is too long to state here, so we omit it.

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